Existence and asymptotics of topological solutions in the self-dual Maxwell-Chern-Simons $O(3)$ sigma model

Jongmin Han

(With Kyungwoo Song)

Hankuk University of Foreign Studies

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1. Introduction

● Classical O(3) Sigma models
(1) The classical O(3) sigma model is described by fields $\phi_1, \phi_2, \phi_3$ under $\sum \phi_i^2 = 1$ and is useful for Heisenberg ferromagnet.
(2) The solution is scale invariant which makes the model unsuitable as models for particles.
(3) Many attempts to break the scale invariance - introduction of gauge fields

● Gauged O(3) Sigma models
(1) Maxwell gauged O(3) Sigma model by [Schroers,1995] etc.- approximate soliton model for heavy particles
(2) Chern-Simons gauged O(3) Sigma model by [Ghosh-Ghosh,1996] etc. - models for fractional statistics
(3) unification - Maxwell-Chern-Simons gauged O(3) Sigma model by [Kimm-Lee-Lee,1996]
• Self-dual equations for Maxwell-Chern-Simons $O(3)$ sigma model (MCS-$O(3)$) [Kimm-Lee-Lee, 1996]

\[
\Delta w = 2q\left(-N + \frac{2a(e^w - 1)}{(a + 1)(e^w + a)}\right) + 4\pi \sum_{j=1}^{l} n_\delta \delta_p, \\
\Delta N = -\kappa^2 q^2 \left(-N + \frac{2a(e^w - 1)}{(a + 1)(e^w + a)}\right) + \frac{4aqe^w N}{(e^w + a)^2}.
\]

• Self-dual equations for Abelian Maxwell-Chern-Simons model (MCS) [Lee-Lee-Min, 1990]

\[
\Delta w = 2q(-N + e^w - 1) + 4\pi \sum_{j=1}^{l} n_\delta \delta_p, \\
\Delta N = -\kappa^2 q^2 (-N + e^w - 1) + 2qe^w N.
\]

• Constants and Topological boundary condition: \( \kappa, q > 0, a \geq 1 \) and

\[ w, N \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad \mathbb{R}^2. \]
- Unified form

\[ \Delta w = 2q(-N + f(e^w)) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \]
\[ \Delta N = -\kappa^2 q^2 (-N + f(e^w)) + 2q f'(e^w) e^w N. \]

(i) MCS-O(3)

\[ f(t) = \frac{2a(t - 1)}{(a + 1)(t + a)} \]

(ii) MCS

\[ f(t) = t - 1 \]
• Maxwell limit (M-lim): $\kappa \to 0$ and $N \equiv 0$

$$\Delta w = 2qf(e^w) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}.$$ 

(i) MCS $\rightarrow$ Abelian-Higgs model: [Jaffe-Taubes, 1980]

(ii) MCS-O(3) $\rightarrow$ Maxwell gauged O(3) sigma model (M-O(3)): [Schroers, 1996], [Yang, 1997]
• Chern-Simons limit (CS-lim)

From

\[ \Delta w = 2q(-N + f(e^w)) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \]

\[ \Delta N = q \left( -\kappa^2 q(-N + f(e^w)) + 2f'(e^w)e^w N \right), \]

we formally derive

\[ N = f(e^w), \quad q(-N + f(e^w)) = \frac{2}{\kappa^2} f'(e^w)e^w N = \frac{2}{\kappa^2} f'(e^w)e^w f(e^w). \]

Hence,

\[ \Delta w = \frac{4}{\kappa^2} f(e^w) f'(e^w)e^w + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}. \]

(i) MCS → Abelian Chern-Simons-Higgs model : [Hong-Kim-Pac, 1990], [Jackiw-Weinberg, 1990], D. Chae, R. Wang, Y. Yang, C.-S. Lin, J. Han, K. Choe, G. Trantello, M. Nolasco, etc.

(ii) MCS-O(3) → Chern-Simons gauged O(3) sigma model (CS-O(3)) : [Kimm-Lee-Lee, 1996], [Schroers, 1996], [Yang, 1997], [Choe-Nam, 2007]
• **Problems:** existence, proof of M-limit, proof of CS-limit.

• **Known Results for MCS by [Chae-Kim, 1997]**

(1) existence of variational solutions

(2) existence of maximal solutions given by iterative sequence \((w_j, N_j)\) - convergence is obtained by the monotonicity \((w_j, N_j) \geq (w_{j+1}, N_{j+1})\) with \(G(w_j, N_j) \geq G(w_{j+1}, N_{j+1})\) where \(G\) is a functional associated with the eqns.

(3) M-limit for arbitrary solutions: \((w_\kappa, N_\kappa) \rightarrow (w_M, 0)\) by use of the monotonicity of \(G\) together with \((w_\kappa, N_\kappa) \leq (w_M, 0)\) which leads to \(G(w_\kappa, N_\kappa) \leq G(W_M, 0)\) but requires an iteration scheme.

(iii) CS-limit for maximal solutions: \((w_q, N_q) \rightarrow (w_{CS}, f(e^{w_{CS}}))\) by monotonicity of \((w_q, N_q)\) with respect to \(q\), i.e., \((w_q, N_q) \leq (w_{q'}, N_{q'})\) if \(q \leq q'\), which also needs an iteration scheme.
• Known Results for MCS-O(3) by [H.-Nam, 2005]
  There exists a constant $\kappa_0$ satisfying that for each $0 < \kappa < \kappa_0$, there is a constant $q_\kappa > 0$ such that the equations admit a maximal solution $(w, N) \in C^\infty(\mathbb{R}^2 \setminus \{p_j\}) \times C^\infty(\mathbb{R}^2)$ for all $q > q_\kappa$. Moreover, the functions $w^2, N^2, |\nabla w|^2, |\nabla N|^2$ decay exponentially at the infinity.

• Remark: Constraints on $\kappa$ and $q$ come from the iteration and finding a subsolution.
• **Purpose**: existence of topological MCS-O(3) solutions for arbitrary $\kappa$ and $q$, proof of M-limit and CS-limit.

• **Other results for M-lim and CS-lim**
  (1) [Chae-Kim, 1997; Ricciardi-Tarantello, 2002]: M-lim, CS-lim for periodic solutions of MCS
  (2) [Ricciardi, 2003]: CS-lim for periodic solutions of MCS-O(3)
  (3) [H.-Jang, 2005]: CS-lim for nontopological bare solutions of MCS
Theorem 1. For any \( \kappa, q > 0 \), there exists a solution of MCS-O(3).

Theorem 2. For fixed \( q > 0 \), let \((u_\kappa, N_\kappa)\) be any solution pair of MCS-O(3) corresponding to \( \kappa \). Then, there exists a function \( u^* \) such that

\[
\|u_\kappa - u^*\|_{H^k(\mathbb{R}^2)}, \quad \|N_\kappa\|_{H^k(\mathbb{R}^2)} \to 0
\]

for any nonnegative integer \( k \) as \( \kappa \to 0 \). Furthermore, \( u^* \) is the unique solution of M-O(3).

Theorem 3. Let \( \kappa > 0 \) be fixed. Given \( q > 0 \), let \((u_q, N_q)\) be a solution of MCS-O(3) obtained by Theorem 1. Then, as \( q \to \infty \), there exist a subsequence, still denoted by \((u_q, N_q)\), and a pair of functions \((u_*, N_*)\) such that

\[
\|u_q - u_*\|_{C^k(K)}, \quad \|N_q - N_*\|_{C^k(K)} \to 0
\]

for any nonnegative integer \( k \) and compact sets \( K \subset \mathbb{R}^2 \). Moreover, \( u_* \) is a solution of CS-O(3) and \( N_* = f(e^{u_*}) \).
2. Existence

- Regularization

To remove the singular terms, we introduce a reference function 

$$v_{0,\mu}(x) = \sum_{j=1}^{l} n_j \ln \left( \frac{|x - p_j|^2}{\mu + |x - p_j|^2} \right), \quad \mu \geq 1.$$ 

Then,

$$\Delta v_{0,\mu} = -g_\mu + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \quad g_\mu(x) = \sum_{j=1}^{l} \frac{4\mu n_j}{(\mu + |x - p_j|^2)^2}.$$ 

If we set $$v = w - v_{0,\mu},$$ then we have

$$\Delta v = 2q \left( -N + f(e^{v_{0,\mu} + v}) \right) + g_\mu,$$

$$\Delta N = -\kappa^2 q^2 \left( -N + f(e^{v_{0,\mu} + v}) \right) + 2q f'(e^{v_{0,\mu} + v}) e^{v_{0,\mu} + v} N,$$

with the boundary conditions

$$v, N \to 0 \text{ as } |x| \to \infty.$$
• Fourth order equation

Solving for $N$ from the 2nd eqn, we obtain

$$N = -\frac{1}{2q} \Delta v + f(e^{v_0,\mu} + v) + \frac{1}{2q} g_\mu.$$ 

Then substituting this into the 1st eqn, we have a fourth order equation

$$\begin{cases}
\Delta^2 v - \kappa^2 q^2 \Delta v + 4q^2 f(e^{v_0,\mu} + v)f'(e^{v_0,\mu} + v)e^{v_0,\mu} + v - 2q \Delta f(e^{v_0,\mu} + v) \\
\quad - 2q f'(e^{v_0,\mu} + v)e^{v_0,\mu} + v(\Delta v + \Delta v_0,\mu) + (\kappa^2 q^2 g_\mu - \Delta g_\mu) = 0, \\
v \to 0 \text{ as } |x| \to \infty.
\end{cases}$$

• Functional

$$\mathcal{F}_\mu(v) = \int_{\mathbb{R}^2} \frac{1}{2} |\Delta v|^2 + \frac{1}{2} \kappa^2 q^2 |\nabla v|^2 + 2q^2 f^2(e^{v_0,\mu} + v) + 2q f'(e^{v_0,\mu} + v)e^{v_0,\mu} + v|\nabla (v_0,\mu + v)|^2 + (\kappa^2 q^2 g_\mu - \Delta g_\mu) v$$
• **Theorem 1'.** There exists $\mu_0$ such that $F_\mu$ has a minimizer on $H^2(\mathbb{R}^2)$ for all $\mu > \mu_0$. Here, $\mu_0$ is independent of $\kappa$ and $q$.

**Proof.** Since $F_\mu$ is weakly lower semi-continuous, it suffices to prove that $F_\mu$ is coercive. Note that

$$F_\mu(v) \geq \int_{\mathbb{R}^2} \frac{1}{2} |\Delta v|^2 + \frac{1}{2} \kappa^2 q^2 |\nabla v|^2 + 2q^2 f^2(e_{v_0,\mu} + v) + (\kappa^2 q^2 g_\mu - \Delta g_\mu)v.$$

Note that

$$2 \int_{\mathbb{R}^2} \frac{4a^2}{(a + 1)^2} \left( \frac{e_{v_0,\mu} + v - 1}{e_{v_0,\mu} + v + a} \right)^2 \geq c_1 \int_{\mathbb{R}^2} \frac{e^{2v_0,\mu}(e^v - 1)^2}{2(e_{v_0,\mu} + v + a)^2} - c_1 \int_{\mathbb{R}^2} \frac{(e_{v_0,\mu} - 1)^2}{(e_{v_0,\mu} + v + a)^2}$$

$$\geq c_1 \int_{\mathbb{R}^2} \frac{e^{2v_0,\mu}(e^v - 1)^2}{2(e_{v_0,\mu} + v + a)^2} - c_{0,\mu}$$

$$= c_1 \left( \int_{\{v \leq \ln T\}} + \int_{\{v \geq \ln T\}} \right) - c_{0,\mu}$$

$$\geq c_2 \int_{\mathbb{R}^2} \frac{v^2 e^{2v_0,\mu}}{(1 + |v|)^2} - c_{0,\mu},$$

where $T > 1$ is a number such that $t \geq (1 + a) \ln t + 1$ for all $t \geq T$ and we used the inequality $|e^t - 1| \geq |t|/(1 + |t|)$ for $t \in \mathbb{R}$. 

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We decompose
\[ 2 \int_{\mathbb{R}^2} f^2(e^{v_0+v}) \geq c_2 \int_{\mathbb{R}^2} \frac{v^2 e^{2v_0,\mu}}{1 + |v|^2} - c_{0,\mu} = c_2 \left( \int_{\Omega_\delta} + \int_{\Omega_{1,\mu} \setminus \Omega_\delta} + \int_{\Omega_{2,\mu}} \right) - c_{0,\mu}, \]
where \(0 < \delta < \min_{i \neq j} \{|p_i - p_j|/2, 1\}\) and
\[ \Omega_\delta = \bigcup_{j=1}^l B_\delta(p_j), \quad \Omega_{1,\mu} = \{ x \in \mathbb{R}^2 : e^{2v_0,\mu} < 1/64 \}, \quad \Omega_{2,\mu} = \{ x \in \mathbb{R}^2 : e^{2v_0,\mu} \geq 1/64 \}. \]
Note that if \(\mu\) is large, then \(|x|^2 \leq c_3(\mu + 1)\) for all \(x \in \Omega_{1,\mu}\). For \(x \in B_\delta(p_j)\), \(e^{-2v_0,\mu} \leq c_4 \mu^{2d} |x - p_j|^{-4d}\), where \(d = \sum n_j\). So, if we choose a number \(\alpha \in (0, 1/(2d + 1))\) such that \(4d\alpha/(1 - \alpha) < 2\), then
\[ \left( \int_{\Omega_\delta} + \int_{\Omega_{1,\mu} \setminus \Omega_\delta} \right) e^{-2v_0,\mu}_{1-\alpha} \leq c_5(\mu^{1+4d\alpha/(1-\alpha)} + 1). \]
It follows from the Hölder inequality that
\[ \int_{\Omega_{1,\mu}} \frac{v^2}{(1 + |v|)^2} \leq \left( \int_{\Omega_{1,\mu}} e^{-2v_0,\mu}_{1-\alpha} \right)^{1-\alpha} \left( \int_{\Omega_{1,\mu}} \frac{e^{2v_0,\mu} v^2}{(1 + |v|)^2} \right)^\alpha \leq C(\mu) + \int_{\Omega_{1,\mu}} \frac{e^{2v_0,\mu} v^2}{(1 + |v|)^2}. \]
Hence,

\[ 2 \int_{\mathbb{R}^2} f^2(e^{v_0} + v) \geq \frac{c_2}{64} \int_{\mathbb{R}^2} \frac{v^2}{(1 + |v|)^2} - C(\mu). \]

We also find that

\[ \int_{\mathbb{R}^2} (\kappa^2 q^2 g_\mu - \Delta g_\mu)v \geq -\frac{c_9 \kappa^2 q^2}{\sqrt{\mu}} \|v\|_2 - \frac{1}{4} \|\Delta v\|_2^2 - C(\mu). \]

From the inequality

\[ \|v\|_2 \leq 2 + \int_{\mathbb{R}^2} \frac{v^2}{(1 + |v|)^2} + 2 \int_{\mathbb{R}^2} |\nabla v|^2, \]

we deduce

\[ \mathcal{F}_\mu(v) \geq \frac{1}{4} \|\Delta v\|_2^2 + \kappa^2 q^2 \left( \frac{1}{2} - \frac{2c_9}{\sqrt{\mu}} \right) \|\nabla v\|_2^2 + q^2 \left( \frac{c_2}{64} - \frac{c_9 \kappa^2}{\sqrt{\mu}} \right) \int_{\mathbb{R}^2} \frac{v^2}{(1 + |v|)^2} - C(\mu). \]

So, if \( \mu \) is large, \( \mathcal{F}_\mu \) is coercive. \( \square \)
3. Maxwell limit

For a fixed $q > 0$, we take $\mu > \max\{\mu_0, d(1 + a)/q\}$ and write $v_0 = v_{0,\mu}$, $g = g_\mu$.

Let $(v_\kappa, N_\kappa)$ be a solution pair corresponding to $\kappa$.

• Theorem 2'. As $\kappa \to 0$, we have
  \[ \|N_\kappa\|_{H^k(\mathbb{R}^2)}, \quad \|v_\kappa - v^*\|_{H^k(\mathbb{R}^2)} = O(\kappa^\alpha) \]
  for some constant $\alpha \in (0, 1/(d + 1))$ and for any nonnegative integer $k$. Here, $v^*$ is the unique solution of the regularized M-O(3) eqn
  \[ \Delta v = 2qf(e^{v_0+v}) + g \]
  and satisfies $e^{v_0+v^*} \leq 1$. 

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• **Lemma** (Maximum Principle)

\[
e^{v_0+v} \leq 1, \quad N \leq 0, \quad -N + f(e^{v_0+v}) \leq 0,
0 \geq N \geq f(e^w) = \frac{2a(e^w - 1)}{(a + 1)(e^w + a)} \geq -\frac{2}{a + 1}.
\]

**Step 1.** \(\|N_\kappa\|_{L^\infty(\mathbb{R}^2)} = O(\kappa^2), \quad \|v_\kappa\|_{L^\infty(\mathbb{R}^2)} = O(1)\)

It follows from the maximum principle that \(\|N_\kappa\|_{L^\infty(\mathbb{R}^2)} = O(\kappa^2)\) and \(v_\kappa \leq v^*\). If \(y_\kappa\) be a minimum point of \(v_\kappa\) such that \(v_\kappa(y_\kappa) \to -\infty\) as \(\kappa \to 0\), then by the maximum principle,

\[2qf(e^{(v_0+v_\kappa)}(y_\kappa)) + g(y_\kappa) \geq 2qN_\kappa(y_\kappa).
\]

If \(\kappa \to 0\), then

\[0 \leq 2qf(0) + \frac{4d}{\mu} = -\frac{4q}{1 + a} + \frac{4d}{\mu} < 0,
\]

which is a contradiction. Here, the last inequality comes from the choice of \(\mu\). As a consequence, we see that \(\|v_\kappa\|_{L^\infty(\mathbb{R}^2)} = O(1)\) as \(\kappa \to 0\).
Step 2. $\|N_\kappa\|_{H^2(\mathbb{R}^2)} = O(\kappa^2)$.

Integrating MCS-O(3), we have

$$2q \int_{\mathbb{R}^2} f'(e^{v_0 + v_\kappa})e^{v_0 + v_\kappa} N_\kappa = -2\pi d\kappa^2 q,$$

which implies that

$$\|N_\kappa\|_{L^1(\mathbb{R}^2)}, \|N_\kappa\|_{L^2(\mathbb{R}^2)}, \|\Delta N_\kappa\|_{L^2(\mathbb{R}^2)} = O(\kappa^2).$$

Hence, $\|N_\kappa\|_{H^2(\mathbb{R}^2)} = O(\kappa^2)$. 
Step 3. \[ \|v_\kappa - v^*\|_{H^2(\mathbb{R}^2)} = O(\kappa^\alpha) \] for some \( \alpha \in (0, 1/(d+1)) \).

We have

\[ \Delta (v^* - v_\kappa) = 2q N_\kappa + 2q \left( f(e^{v_0 + v^*}) - f(e^{v_0 + v_\kappa}) \right). \]

Multiplying this equation by \((v^* - v_\kappa)\), we derive

\[ -\int_{\mathbb{R}^2} N_\kappa (v^* - v_\kappa) \geq \int_{\mathbb{R}^2} \left( f(e^{v_0 + v^*}) - f(e^{v_0 + v_\kappa}) \right) (v^* - v_\kappa) \]
\[ \geq \frac{2ae^{-C_0}}{(1 + a)^2} \int_{\mathbb{R}^2} e^{v_0}(v^* - v_\kappa)^2 \]
\[ \geq \frac{2ae^{-C_0}}{(1 + a)^2} \left( \int_{\Omega_\delta} + \int_{\mathbb{R}^2 \setminus \Omega_\delta} \right) = \frac{2ae^{-C_0}}{(1 + a)^2} (I + II). \]

Note that \( II \geq C \int_{\mathbb{R}^2 \setminus \Omega_\delta} (v^* - v_\kappa)^2 \) and

\[ \int_{\Omega_\delta} (v^* - v_\kappa)^2 \leq C \left( \int_{\Omega_\delta} e^{-v_0 \frac{\alpha}{1-\alpha}} \right)^{1-\alpha} \left( \int_{\Omega_\delta} e^{v_0}(v^* - v_\kappa)^2 \right)^\alpha = CI^\alpha. \]
Hence,
\[
C_1 \left[ \left( \int_{\Omega_\delta} (v^* - v_\kappa)^2 \right)^{\frac{1}{\alpha}} + \int_{\mathbb{R}^2 \setminus \Omega_\delta} (v^* - v_\kappa)^2 \right] \leq - \int_{\mathbb{R}^2} N_\kappa (v^* - v_\kappa)
\]
\[
\leq \frac{C_1}{2} \left[ \left( \int_{\Omega_\delta} (v^* - v_\kappa)^2 \right)^{\frac{1}{\alpha}} + \int_{\mathbb{R}^2 \setminus \Omega_\delta} (v^* - v_\kappa)^2 \right] + C_2 \kappa^2,
\]
which implies that \( \|v^* - v_\kappa\|_2 \leq C \kappa^\alpha \). Moreover,
\[
\|\Delta (v^* - v_\kappa)\|_2 \leq C(\|N_\kappa\|_{L^2(\mathbb{R}^2)} + \|v^* - v_\kappa\|_2) \leq C \kappa^\alpha.
\]

**Step 4.** For higher norms, use induction.
4. Chern-Simons limit

Let $\kappa > 0$ and $\mu > \mu_0$ be fixed. We write $v_0 = v_{0,\mu}$ and $F = F_\mu$.

Given $q > 0$, let $(v_q, N_q)$ be solutions of MCS-O(3) given by Theorem 1.

**Theorem 3’**. There is a subsequence, still denoted by $(v_q, N_q)$, and a pair of functions $(v_*, N_*)$ such that

$$(v_q, N_q) \to (v_*, N_*) \text{ in } C^k_{\text{loc}}(\mathbb{R}^2) \times C^k_{\text{loc}}(\mathbb{R}^2)$$

for any nonnegative integer $k$. Furthermore, $v_*$ is a solution of the regularized CS-O(3) eqn

$$\Delta v = \frac{4}{\kappa^2} e^{v_0+v} f(e^{v_0+v}) f'(e^{v_0+v}) + g$$

and $N_* = f(e^{v_0+v})$. 


Step 1. \( \|v_q\|_{L^\infty(\mathbb{R}^2)} , \|v_q\|_{H^1(\mathbb{R}^2)} < C. \)

For given test function \( \xi \), \( \mathcal{F}(v_q) \leq \mathcal{F}(\xi) \) such that

\[
\int_{\mathbb{R}^2} \left( |\nabla v_q|^2 + \frac{v^2}{(1 + |v_q|)^2} \right) \leq C,
\]

which implies that \( \|v_q\|_{H^1(\mathbb{R}^2)} \leq C. \)

Note that

\[
\Delta \left( v_q + \frac{2}{\kappa^2 q} N_q \right) = g + \frac{4}{\kappa^2} f'(e^{v_0 + v_q})e^{v_0 + v_q} N_q \quad \in \quad L^\infty(\mathbb{R}^2),
\]

uniformly. Hence, \( \|v_q\|_{L^\infty(\mathbb{R}^2)} \leq C. \)
Step 2.

Let us define

\[
\begin{align*}
\varphi_q &= q(-N_q + f(U e^v_0)), \\
\psi_q &= q(-\kappa^2 \varphi_q + 2 f'(U e^v_0) U e^v_q N_q),
\end{align*}
\]

Then,

\[
\Delta v_q = 2 \varphi_q + g, \quad \Delta N_q = \psi_q.
\]

Let \( U = e^v_0 \). We have

\[
\frac{1}{q} \Delta \varphi_q = \left( \kappa^2 q + 2 f'(U e^v_0) U e^v_q \right) \varphi_q + f''(U e^v_0) e^{2v_q} |\nabla U + U \nabla v_q|^2 \\
+ f'(U e^v_0) \left\{ -2 q U e^v_q N_q + e^v_q \Delta U + 2 e^v_q \nabla U \cdot \nabla v_q + U e^v_q |\nabla v_q|^2 + g U e^v_q \right\}
\]

\[
\equiv \left( \kappa^2 q + 2 f'(U e^v_0) U e^v_q \right) \varphi_q + \sigma_q.
\]
Moreover,

\[
\frac{1}{q} \Delta \psi_q = \left( \kappa^2 q + 2 f'(U e^{v_q}) U e^{v_q} \right) \psi_q + 2 f'''(U e^{v_q}) U e^{3v_q} N_q |\nabla U + U \nabla v_q|^2 \\
+ f''(U e^{v_q}) e^{2v_q} \left\{ (4N_q - \kappa^2 q) |\nabla U + U \nabla v_q|^2 + 4U \nabla N_q \cdot (\nabla U + U \nabla v_q) \\
+ 2U N_q (\Delta U + 2 \nabla U \cdot \nabla v_q + 2U \varphi_q + gU + U |\nabla v_q|^2) \right\} \\
+ f'(U e^{v_q}) e^{v_q} \left\{ (2 - \kappa^2 q) \varphi_q + 4 \nabla N_q \cdot (\nabla U + U \nabla v_q) \\
+(2N_q - \kappa^2 q)(\Delta U + 2 \nabla U \cdot \nabla v_q + gU + U |\nabla v_q|^2) \right\} \\
\equiv \left( \kappa^2 q + 2 f'(U e^{v_q}) U e^{v_q} \right) \psi_q + \eta_q.
\]
Lemma Suppose that $-\Delta u = f$ in $\Omega \subset \mathbb{R}^n$. Then,
\[
|\nabla u(x)|^2 \leq C \|u\|_{L^\infty(\Omega)} \left( \|f\|_{L^\infty(\Omega)} + \frac{1}{\text{dist}(x, \partial \Omega)} \|u\|_{L^\infty(\Omega)} \right)
\]
for $x \in \Omega$. Here, $C$ depends only on $\Omega$.

Step 3. $\|\nabla v_q\|_{L^\infty(\mathbb{R}^2)}$, $\|\nabla N_q\|_{L^\infty(\mathbb{R}^2)}$, $\|\varphi_q\|_{L^\infty(\mathbb{R}^2)}$, $\|\psi_q\|_{L^\infty(\mathbb{R}^2)} \leq C$.

By Lemma, $\|\nabla v_q\|_{L^\infty(\mathbb{R}^2)}^2 \leq C(q + 1)$ from MCS-O(3). Hence,
\[
\|\varphi_q\|_\infty \leq \left\| \frac{\sigma_q}{\kappa^2 q + 2 f'(U e^{v_q}) U e^{v_q}} \right\| \leq C.
\]
Returning to MCS-O(3), we see that $\|\nabla v_q\|_{L^\infty(\mathbb{R}^2)}^2 \leq C$. 

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By Lemma, \( \| \nabla N_q \|_{L^\infty(\mathbb{R}^2)}^2 \leq C(q + 1) \) from MCS-O(3). Hence,

\[
\| \psi_q \|_{L^\infty(\mathbb{R}^2)} \leq \left\| \frac{\eta_q}{\kappa^2 q + 2 f'(U e^v_q) U e^v_q} \right\|_{L^\infty(\mathbb{R}^2)} \leq C.
\]

Going back to MCS-O(3), we see that \( \| \nabla N_q \|_{L^\infty(\mathbb{R}^2)}^2 \leq C \).

**Step 4.** By Step 4, we can extract a convergent subsequence in \( C^1_{loc}(\mathbb{R}^2) \). The convergence of higher norms follows from induction. \( \square \)