Blow-up of Electric Fields between Closely Spaced Spherical Perfect Conductors

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Gradient estimate

\[
\begin{aligned}
&\nabla \cdot \left( 1 + \sum_{i=1,2} (k_i - 1) \chi(B_i) \right) \nabla u = 0 \quad \text{in } \mathbb{R}^n, \ n = 2, 3 \\
&u(X) - H(X) = O(|X|^{-n+1}) \quad \text{as } |X| \to \infty.
\end{aligned}
\]

Get the optimal bounds for \( \|\nabla (u - H)\|_{L^\infty} \) in terms of \( \epsilon := \text{dist}(B_1, B_2) \) when \( \epsilon \) is small.

This problem arises in relation with the computation of electromagnetic fields in the presence of fibers and estimates of the stress in composite materials.
Related Results

- **Boundedness:** $0 < k_i < \infty$
  
  (1) E. Bonnetier and M. Vogelius, 2000, touching inclusions in 2D.

  (2) Y.Y. Li and M. Vogelius, 2000, $\nabla (u - H)$ is bounded independently of the distance between the inclusions in two or three dimensions (arbitrary number of inclusions with arbitrary shape).

  (3) Y.Y. Li and L. Nirenberg, 2003, extended (2) to elliptic systems.

- **Unboundedness for** $k = \infty$
  
  (1) B. Budiansky and G.F. Carrier, 1984, showed that the gradient generally becomes unbounded as the distance $\epsilon$ approaches zero in 2D.
Overview

1. Circular inclusions
2. Decomposition of $u$ into blow-up and non-blow-up terms
3. Spherical inclusions
The solution is represented as

\[ u(X) = H(X) + S_{B_1} \varphi_1(X) + S_{B_2} \varphi_2(X) \]

where \( \varphi_i \in L^2_0(\partial B_i), \ i = 1, 2, \) is the unique solution to the system of integral equations

\[
\begin{cases}
\lambda_1 \varphi_1 - \frac{\partial}{\partial v(1)} S_{B_2} \varphi_2 = \frac{\partial H}{\partial v(1)} & \text{on } \partial B_1 \\
\lambda_2 \varphi_2 - \frac{\partial}{\partial v(2)} S_{B_1} \varphi_1 = \frac{\partial H}{\partial v(2)} & \text{on } \partial B_2, \quad (\lambda_i = \frac{k_i+1}{2(k_i-1)}).
\end{cases}
\]

The potentials are given as

\[
\begin{cases}
\varphi_1 &= \frac{1}{\lambda_1} \frac{\partial H}{\partial v(1)} + \frac{1}{\lambda_1} \frac{\partial}{\partial (1)} S_{B_2} \left( \frac{1}{\lambda_2} \frac{\partial H}{\partial v(2)} \right) + \frac{1}{\lambda_1} \frac{\partial}{\partial (1)} S_{B_2} \left[ \frac{1}{\lambda_2} \frac{\partial}{\partial (2)} S_{B_1} \left( \frac{1}{\lambda_1} \frac{\partial H}{\partial v(1)} \right) \right] + \ldots \\
\varphi_2 &= \frac{1}{\lambda_2} \frac{\partial H}{\partial v(2)} + \frac{1}{\lambda_2} \frac{\partial}{\partial (2)} S_{B_1} \left( \frac{1}{\lambda_1} \frac{\partial H}{\partial v(1)} \right) + \frac{1}{\lambda_2} \frac{\partial}{\partial (2)} S_{B_1} \left[ \frac{1}{\lambda_1} \frac{\partial}{\partial (1)} S_{B_2} \left( \frac{1}{\lambda_2} \frac{\partial H}{\partial v(2)} \right) \right] + \ldots
\end{cases}
\]
\[ \frac{\partial u}{\partial \nu} \bigg|_\pm = (\lambda \pm \frac{1}{2}) \varphi_i, \quad \text{on } \partial B_i, \ i = 1, 2. \]

\( \varphi_i \)'s can be expressed as absolutely convergent series:

\[
\begin{cases}
\varphi_1 = \frac{1}{\lambda_1} \sum_{k=0}^{\infty} \frac{1}{(4\lambda_1\lambda_2)^k} \frac{\partial}{\partial \nu} \left[ (R_2R_1)^k(I - \frac{1}{2\lambda_2} R_2)H \right] \bigg|_{\partial B_1}, \\
\varphi_2 = \frac{1}{\lambda_2} \sum_{k=0}^{\infty} \frac{1}{(4\lambda_1\lambda_2)^k} \frac{\partial}{\partial \nu} \left[ (R_1R_2)^k(I - \frac{1}{2\lambda_1} R_1)H \right] \bigg|_{\partial B_2},
\end{cases}
\]

where

\[ R_i(X) := \frac{r_i^2(X - Z_i)}{|X - Z_i|^2} + Z_i, \quad i = 1, 2. \]

\[ R_i f(x) := f(R_i(x)) \]

Observe that

\[ \nabla((R_2R_1)^k H)(X) = \nabla H((R_1R_2)^k X) \prod_{i=1}^{2k} DR_{l_i}(R_{l_{i-1}} \cdots R_{l_1}(X)). \]
(1/\epsilon)-bound

Proof.

Observe that

\[
\nabla((R_2R_1)^kH)(X) = \nabla H((R_1R_2)^kX) \prod_{i=1}^{2k} DR_{l_i}(R_{l_{i-1}} \cdots R_{l_1}(X)).
\]

From the fact

\[
|DR_j(X)| \leq \frac{r_j^2}{|X - z_j|^2} \leq \frac{1}{(1 + \frac{\epsilon}{r_j})^2}, \quad x \in \mathbb{R}^2 \setminus B_j, \quad j = 1, \ldots, m,
\]

the series absolutely converges and

\[
|\varphi_1| \leq C \frac{\min(r_1, r_2)}{\epsilon}
\]

for a constant C independent of \(k_1, k_2, r_1\) and \(r_2\).
Where the reflected points are located?

After some reflections, boundary points of the balls are $\frac{1}{\sqrt{\epsilon}}$ away from the middle point of two balls. So we have $\frac{1}{\sqrt{\epsilon}}$ bounds.
Let $X_j$ be the point on $\partial B_j$ closest to other disk, and let $X^c$ be the middle point of $X_1$ and $X_2$, and $I$ is the line segment connecting $X_1$ and $X_2$.

Define $r_* := \sqrt{(2r_1r_2)/(r_1+r_2)}$ and $\tau := 1/(4\lambda_1\lambda_2)$ ( $\lambda_l = \frac{k_l+1}{2(k_l-1)}$).

(i) If $k_1, k_2 > 1$

$$|\nabla u|_+(X_j) \geq C \frac{\inf_{x \in I} |\nabla H(x) \cdot N|}{1 - \tau + (r_*/r_{\min}) \sqrt{\epsilon}}, \quad j = 1, 2.$$ 

If $k_1, k_2 < 1$,

$$|\nabla u|_+(X_j) \geq C \frac{\inf_{x \in I} |\nabla H(x) \cdot T|}{1 - \tau + (r_*/r_{\min}) \sqrt{\epsilon}}, \quad j = 1, 2.$$ 

(ii) Then there is a constant $C_2$ independent of $k_1, k_2, r_1, r_2, \epsilon$, and $\Omega$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{1 - |\tau| + (r_*/r_{\max}) \sqrt{\epsilon}}.$$
Blow-up when $k_i = \infty$
Suppose that the conductivity of $\Omega$ is 1 and that of $B$ is $k \neq 1$. We consider the following Dirichlet and Neumann problems: for a given $f \in C^{1,\alpha}(\partial \Omega)$, $\alpha > 0$,

\[
\begin{cases}
\nabla \cdot (1 + (k - 1)\chi(B))\nabla u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega,
\end{cases}
\]

and for a given $g \in C^{\alpha}(\partial \Omega)$

\[
\begin{cases}
\nabla \cdot (1 + (k - 1)\chi(B))\nabla u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = g & \text{on } \partial \Omega.
\end{cases}
\]
For a given $f \in C^{1,\alpha}(\partial \Omega)$, $\alpha > 0$,

$$\begin{cases}
\nabla \cdot (1 + (k - 1)\chi(B)) \nabla u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega,
\end{cases}$$

$$u(X) = D_\Omega(f)(X) - S_\Omega(g)(X) + S_B(\varphi)(X), \quad X \in \Omega|_{\partial \Omega},$$

where $g, \varphi \in L^2_0(\partial B)$ satisfies where

$$\begin{cases}
\frac{1}{2} g - \frac{\partial (S_B \varphi)}{\partial n_\Omega} = \frac{\partial (D_\Omega f)}{\partial n_\Omega} & \text{on } \partial \Omega, \\
\lambda \varphi + \frac{\partial (S_\Omega g)}{\partial n_B} = \frac{\partial (D_\Omega f)}{\partial n_B} & \text{on } \partial B.
\end{cases}$$

with $\lambda = \frac{k + 1}{2(k - 1)}$. 

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Blow-up of Electric Fields
For the Dirichlet problem we have the following theorem. Let
\[ \epsilon := \text{dist}(B, \partial\Omega), \quad \sigma := \frac{k - 1}{k + 1}, \quad r^* := \sqrt{\frac{\rho - r}{\rho r}}, \]
and let \( u \) be the solution to Dirichlet problem.

(i) If \( k > 1 \), then there exist constants \( C_1 \) independent of \( k, r, \epsilon, \) and \( f \) such that for \( \epsilon \) small enough,
\[
|\nabla u|_+(X_1) \geq C \frac{\left| \frac{\partial D_\Omega(f)}{\partial \nu_B}(X_1) \right| + \| D_\Omega(f) \|_{C^2(\Omega)} O(\sqrt{\epsilon})}{1 - \sigma + 4r^* \sqrt{\epsilon}},
\]
and
\[
|\nabla u|_-(X_2) \geq C \frac{\left| \frac{\partial D_\Omega(f)}{\partial \nu_B}(X_2) \right| + \| D_\Omega(f) \|_{C^2(\Omega)} O(\sqrt{\epsilon})}{1 - \sigma + 4r^* \sqrt{\epsilon}}.
\]

Here \( \nu_B \) and \( \nu_\Omega \) denote the outward unit normal to \( \partial B \) and \( \partial \Omega \). Here \( X_1 \) be the point on \( \partial B \) closest to \( \partial \Omega \) and \( X_2 \) be the point on \( \partial \Omega \) closest to \( \partial B \).

(ii) For any \( k \neq 1 \), there exists a constant \( C_2 \) independent of \( k, r, \) and \( \epsilon \) such that for \( \epsilon \) small enough,
\[
\| \nabla u \|_{L^\infty(\Omega)} \leq C_2 \| f \|_{C^{1,\alpha}(\partial \Omega)} \frac{1}{1 - |\sigma| + r^* \sqrt{\epsilon}}.
\]
Remind that, when $k_i \gg 1$, we have

$$
\frac{C_1 \inf_{x \in N} |\nabla H(x) \cdot N|}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{r_*}{r_{\min}} \sqrt{\epsilon}} \leq \|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{r_*}{r_{\max}} \sqrt{\epsilon}},
$$

Let $H(x) = T \cdot x$, then Blow-up doesn’t happen in LHS, but it does in RHS.
For \( x \in \partial B_1 \), we have

\[
\frac{\partial}{\partial \nu} [(R_2 R_1)^n H](x) = \nu(x) \cdot (\nabla H)((R_1 R_2)^n(x)) \\
\times \prod_{i=0}^{n-1} g_1((R_2 R_1)^i R_2(x))g_2((R_1 R_2)^i(x))
\]

where \( \nu \) is the unit outward normal vector to \( \partial B_1 \).

Investigating where the reflected points are located, for all \( x \in \overline{B}_1 \), we have

\[
\sum_{n=1}^{\infty} \prod_{k=1}^{n} g_1((R_2 R_1)^k(x))g_2((R_1 R_2)^k R_1(x)) \leq \frac{C}{|x - X^c| + \sqrt{\epsilon}},
\]

Suppose \( k_i \gg 1 \), then for the following two cases the gradient is bounded.

- If \( \nabla H(x) = O(|x - X^c|) \)
- \( H(x) = x_2 \).
Suppose that $k_1, k_2 \gg 1$. Let $u_s$ be the solution with $H$ replaced with $H_n(x) = (\nabla H(X^c) \cdot N) (x - X^c) \cdot N$ and let $u_r$ be the solution with $H$ replaced with $H - H_n$ so that $u$ can be decomposed as

$$u = u_s + u_r.$$

Then the following estimates hold. There are constants $C_1, C_2, C_3$ independent of $k_1, k_2, r_1, r_2, \delta$ such that

(i) $|\nabla u_s| + (X_j) \geq \frac{C_1 |\nabla H(X^c) \cdot N|}{k_1^{-1} + k_2^{-1} + \frac{r_*}{r_{\min}} \sqrt{\delta}}, \quad j = 1, 2,$

(ii) $\|\nabla u_s\|_{L^\infty(\Omega)} \leq \frac{C_2 |\nabla H(X^c) \cdot N|}{k_1^{-1} + k_2^{-1} + \frac{r_*}{r_{\max}} \sqrt{\delta}},$

(iii) $\|\nabla u_r\|_{L^\infty(\Omega)} \leq C_3.$
Positive permittivity

For a harmonic function \( H := H_1 + iH_2 \) in \( \mathbb{R}^2 \), let \( u \) be the solution to the following complex conductivity equation:

\[
\begin{cases}
\nabla \cdot ((\sigma + i\omega\varepsilon)\nabla u) = 0 & \text{in } \mathbb{R}^2, \\
u(x) - H(x) = O(|x|^{-1}), & |x| \to \infty,
\end{cases}
\]

where \( \omega \) is the frequency.

We derived the following:

Suppose that \( \sigma_1, \sigma_2 \ll 1 \). There are constants \( C_1, C_2 \) independent of \( \sigma_1, \sigma_2, \epsilon_1, \epsilon_2, r_1, r_2, \delta \), such that

\[
\begin{align*}
(i) & \quad \| \nabla u_s \|_{L^\infty(\Omega \setminus (B_1 \cup B_2))} \leq \frac{C_2 |\nabla H(X^c) \cdot T|}{\sigma_1 + \sigma_2 + \omega(\epsilon_1 + \epsilon_2) + \frac{r^*}{r_{\max}} \sqrt{\delta}}, \\
(ii) & \quad \| \nabla u_r \|_{L^\infty(\Omega \setminus (B_1 \cup B_2))} \leq C_3.
\end{align*}
\]
K. Yun’s Idea for the case of $k = \infty$

Based on this, Yun got optimal bounds valid for convex inclusions.

We consider the following equation

$$
\begin{cases}
\Delta u = 0, & \text{in } \Omega \setminus (D_1 \cup D_2), \\
u|_{\partial D_i} = C_i \text{ (constant)}, & \text{for } i = 1, 2, \\
\int_{\partial D_i} \partial_\nu u \, dS = 0, & \\
u(x) - H(x) = O(|x|^{1-n}) & \text{as } |x| \to \infty.
\end{cases}
$$

Note that

$$
\|\nabla u\|_{L^\infty} \geq \frac{|C_1 - C_2|}{\epsilon}.
$$

Define

$$
\begin{cases}
\Delta h = 0, & \text{in } \mathbb{R}^n \setminus (D_1 \cup D_2), \\
h = O(|x|^{1-n}), & \text{as } |x| \to \infty, \\
h|_{\partial D_i} = k_i \text{ (constant)}, & \text{for } i = 1, 2, \\
\int_{\partial D_i} \partial_\nu h \, ds = (-1)^i + 1, & 
\end{cases}
$$

Then we have that

$$
|u|_{\partial D_1} - |u|_{\partial D_2} = \int_{\partial D_1} (\partial_\nu h) HdS + \int_{\partial D_2} (\partial_\nu h) HdS.
$$
Construct of \( h \) in \( \mathbb{R}^2 \) in the case of circular domains

Let \( p_1 \in D_1 \) and \( p_2 \in D_2 \) be the fixed points of \( R_1(R_2) \) and \( R_2(R_1) \), respectively.

Considering the Apollonius’ Circle and the fact that \( \ln(ab) = \ln a + \ln b \), \( h \) is given by

\[
h = \frac{1}{2\pi} \left( \log |x - p_1| - \log |x - p_2| \right).
\]

For \( H(x_1, x_2) = x_1 \), the solution \( u \) satisfies that

\[
u|_{\partial D_1} - u|_{\partial D_2} = H(p_1) - H(-p_2) \approx 4\sqrt{\frac{r_1r_2}{r_1 + r_2}} \sqrt{\epsilon}.
\]
By Ammari, Dassios, Kang, and Lim in 2007, it is obtained that the gradient doesn’t blow-up if $C_1 = C_2$ for spherical domains.
In $\mathbb{R}^3$ for $k = \infty$, by Y. Y. Li et al 2009

For the solution $u$ to

\[
\begin{align*}
\Delta u &= 0, \quad \text{in } \Omega \setminus (D_1 \cup D_2), \\
\frac{\partial u}{\partial D_i} &= C_i \text{ (constant)}, \\
\int_{\partial D_i} \frac{\partial v}{\partial n} u \, dS &= 0, \quad \text{for } i = 1, 2, \\
u &= \varphi, \quad \text{on } \partial \Omega
\end{align*}
\]

we have

\[
\|\nabla u\|_{L^\infty} \leq C \frac{1}{\epsilon |\ln \epsilon|}.
\]

An important IDEA is that $u$ is decomposed as

\[
u = v_0 + C_1 v_1 + C_2 v_2,
\]

where $v_0$ doesn’t have the potential difference on two conductors, but $v_1$ and $v_2$ have the difference 1.
It is also proved that

\[
\frac{1}{\varepsilon} |C_1 - C_2| \leq \| \nabla u \|_{L^\infty(\Omega \setminus (D_1 \cup D_2))} \leq \frac{C}{\varepsilon} |C_1 - C_2| + C \| \varphi \|_{C^2(\partial \Omega)}.
\]

Now, remaining part is to obtain \( C_1 - C_2 \).

\[
\frac{|Q_\varepsilon[\varphi]|}{C} \leq |C_1 - C_2| \leq C \| \varphi \|_{C^2(\partial \Omega)},
\]

where \( Q_\varepsilon[\varphi] := \int_{\partial D_1} \frac{\partial v_3}{\partial v} \int_{\partial \Omega} \frac{\partial v_2}{\partial v} - \int_{\partial D_2} \frac{\partial v_3}{\partial v} \int_{\partial \Omega} \frac{\partial v_1}{\partial v} \).
Optimal bounds in terms of the radii as well

How the geometrical information of the conductors influences on the blow-up rate?

IDEA: K. Yun’s method to calculate the blow-rate for the general shaped domain in $\mathbb{R}^2$.

Reminded that he used the harmonic function $h$ as follows:

$$
\begin{cases}
\Delta h = 0, & \text{in } \mathbb{R}^n \setminus (D_1 \cup D_2), \\
h = O(|x|^{1-n}), & \text{as } |x| \to \infty, \\
h|_{\partial D_i} = k_i \text{ (constant)}, \\
\int_{\partial D_i} \partial_\nu h \, ds = (-1)^{i+1}, & \text{for } i = 1, 2,
\end{cases}
$$

with the following property:

$$
u|_{\partial D_1} - \nu|_{\partial D_2} = \int_{\partial D_1} (\partial_\nu h)HdS + \int_{\partial D_2} (\partial_\nu h)HdS.
$$
In $\mathbb{R}^3$

We construct

$$h(x) = \frac{1}{(2-n)\omega_n} \frac{1}{M} \left[ Q_2 \sum_{m=1}^{\infty} \frac{(-1)^m (q_{1,m})^{n-2}}{|x - c_{1,m}|^{n-2}} - Q_1 \sum_{m=1}^{\infty} \frac{(-1)^m (q_{2,m})^{n-2}}{|x - c_{2,m}|^{n-2}} \right],$$

where

$$Q_s = \sum_{m=0}^{\infty} \left[ (-1)^m (q_{s,m})^{n-2} \right], \quad s = 1, 2,$$

$$M = Q_2 \sum_{k=0}^{\infty} (q_{1,2k})^{n-2} + Q_1 \sum_{k=0}^{\infty} (q_{(2,2k+1)})^{n-2}.$$

Here $q_{j,k}$ are related with the reflected points of the centers of $B_1$ and $B_2$. 
For \( H(x_1, x_2, x_3) = a_1 x_1 + a_2 x_2 + a_3 x_3 \).

Then, there exists a constant \( C \) independent of \( \epsilon, r_1, r_2 \) and \((a_1, a_2, a_3)\) such that

\[
\frac{1}{C} |a_1| \left( \frac{r_1 r_2}{r_1 + r_2} \right) \frac{1}{|\log \epsilon|} \leq \left| u|_{\partial D_1} - u|_{\partial D_2} \right| \leq C |a_1| \left( \frac{r_1 r_2}{r_1 + r_2} \right) \frac{1}{|\log \epsilon|}
\]

for sufficiently small \( \epsilon > 0 \).
Review: Thank you!

1. Circular inclusions
2. Decomposition of $u$ into blow-up and non-blow-up terms
3. Spherical inclusions